A CLASS OF N NONLINEAR HYPERBOLIC CONSERVATION LAWS

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Abstract

The Riemann problem for a class of nonlinear systems of first order hyperbolic conservation laws is studied. The class consist of systems where the derivative of the flux function, is a lower triangular matrix. In the class there is both strictly and non-strictly hyperbolicity. There are no assumptions on genuine nonlinearity. Existence and uniqueness are proved except in an set with measure zero in the phase space and a set with measure zero in the flux function space where there is a continuum of solutions. An example shows that the solution does not necessarily depend continuously on the data. Numerical methods are discussed.

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1 INTRODUCTION

In this paper we study the Riemann problem for the system of differential equation

$$(1.1) \qquad u_{i_t} + f_i(u_1,\dots,u_i)_x = 0, \qquad i=1,2,\dots,n$$

where f_i is continuous and $\frac{\partial f_i}{\partial u_i}$ is defined except in a finite number of points. Assume further that $\frac{\partial f_i}{\partial u_i}$ is piecewise monotone with a finite number of intervals where the function is monotone. In order to always have a solution it is also necessary with some restrictions on the behaviour of f_i when $|u_i|$ is large.

In the Riemann problem the initial condition is

$$(1.2) \qquad u_i(x,0) = \begin{cases} u_{i,-} & \text{for } x < 0 \\ u_{i,+} & \text{for } x > 0 \end{cases} \qquad i=1,2,\dots,n.$$

In problem (1.1) the matrix $\{ \frac{\partial f_i}{\partial u_j} \}_{i,j}$ is assumed to be lower triangular. The eigenvalues of this matrix are the diagonal elements. The problem is therefore always hyperbolic, and we will call it a lower triangular hyperbolic system. There are no assumptions on the degeneracy of the eigenvalues. Therefore the class to be studied contains both strictly and non-strictly hyperbolic systems. Genuine nonlinearity for this class of problem is equivalent to $\frac{\partial^2 f_i}{\partial u_i^2} \neq 0$, $i=1,\dots,n$. We will allow for loss of genuine nonlinearity in this paper.

For n=1, i.e. the scalar problem, existence and uniqueness are well-known. See e.g. Oleinik [10] and [11] and Smoller [12]. For systems most of results are either for n=2, see e.g. Holden [4], Smoller [13], Keyfitz and Kranzer [6] and [7] or for the strictly hyperbolic case, see e.g. Lax [8].

The Riemann problem is a particular physical problem where it is possible to find an explicit solution. In addition it is used as building blocks in the Cauchy problem with general initial data. In fact, the Riemann problem is used both for existence and uniqueness theorems and as a numerical method. It is used in both ways in the celebrated paper by Glimm [1] and in a paper by Holden, Holden and Høegh-Krohn [3]. Godunov [2] uses the Riemann problem in a numerical method.

There are two main reasons to study a lower triangular hyperbolic system.

First, by restricting ourselves to the analysis of lower triangular systems, we are able to solve the Riemann problem for a large class of nxn systems, including both genuine linearity and nonlinearity and both strictly and non-strictly hyperbolicity. We therefore find some characteristica of the general problem.

Secondly, it is possible to approximate the solution of some physical problems by the solution of (1.1). We will mention two physical problems. In incompressible flow with oil, water and gas, the fractional flow function of gas depends almost only on the gas saturation. In two phase incompressible flow the fractional flow function may change between different rock types. This is modelled in a lower triangular hyperbolic system by letting the first independent variable only depend on the rock type. The solution of the first equation is therefore only a shock with speed zero at the border between the different rock types.

There are no smooth solutions of (1.1) with general initial data except for small t, no matter how smooth the flux function is. Therefore we are interested in weak solutions. There are several smooth solutions of the problem. In this paper we use an entropy critera with travelling waves in order to find the relevant solution. See chapter 24 in [12]. A shock with speed s and with values u_- and u_+ to the left and to the right respectively is deemed admissibly iff there exists an integral curve

(1.3) $$u'(\xi) = f(u(\xi)) - s\, u(\xi) - (\, f(u_\pm) - s\, u_\pm)\ .$$

and $\quad u(\xi) \to u_\pm \quad$ when $\quad \xi \to \pm\infty.$

We name this integral curve an entropy curve in order to separate it from other integral curves.

The origin for the entropy criteria is that the solution is the limiting solution when a second order term vanishes. Assuming a solution of the form $u(x,t) = v(x-st)$ of the regularized equation

$$u_t + f(u)_x = \epsilon\, u_{xx},$$

the limiting solution when ϵ vanishes satisfies the entropy criteria above.

A solution satisfying the entropy condition will also satisfy the jump condition

(1.4) $$s\,(\, u_+ - u_-\,) = f(u_+) - f(u_-).$$

The solution of the Riemann problem $u(s) = u(\frac{x}{t})$ is made up of three types of elementary waves (solutions), namely

(i) constant states,
(ii) shock waves satisfying the entropy condition above and
(iii) rarefaction waves, i.e. continuous solutions satisfying the ordinary diffential equation

$$-\, s\, u_s + f(u)_s = 0.$$

In order to always get a solution we have to accept adjacent shocks with the same speed. This is also necessary in the scalar equation.

In the following chapter we prove existence of a solution of (1.1) and (1.2) for all initial values and uniqueness almost everywhere. Some characteristica of the solution are discussed in chapter 3. We show that the Lax shock inequalities are not always valid for non-strictly hyperbolic systems. An example shows that the solution does not depend continuously on the initial function. Finally some numerical methods are discussed. The general solution depends on the entropy curves. This slows down the speed of the numerical method. When u is varying in the the phase space, the eigenvalues are varying. When eigenvalues are in distinct intervals and f_i is piecewise linear, it is possible to find the solution only with convex/concave envelopes. We may then ignore the entropy curves when finding the correct solution.

2 EXISTENCE AND UNIQUENESS

We will first state the main theorem in the paper.

Theorem 2.1

Assume f is continuous and that $\frac{\partial f_i}{\partial u_i}$ is defined except in a finite number of points. Assume further that $\frac{\partial f_i}{\partial u_i}$ is piecewise monotone and monotone in a finite number of intervals. Let $s_{i,min}$ and $s_{i,max}$ be the smallest and the largest possible speed in a shock in equations $1,\dots,i-1$. Assume that for $i>1$ there exists a $u_{i,min}$ and a $u_{i,max}$ such that

$$\text{either } \frac{\partial f_i}{\partial u_i}(u) < s_{i,min} \text{ or } \frac{\partial f_i}{\partial u_i}(u) > s_{i,max} \quad \text{when } u_i < u_{i,min}$$

and

$$\text{either } \frac{\partial f_i}{\partial u_i}(u) < s_{i,min} \text{ or } \frac{\partial f_i}{\partial u_i}(u) > s_{i,max} \quad \text{when } u_i > u_{i,max}.$$

There exists a solution to the Riemann problem (1.1) and (1.2). The solution is unique except for functions f in a set with measure zero and for a given f inital values u_- and u_+ in a set with measure zero in the (u_-,u_+) plane. There is always uniqueness if $n<2$.

The set of flux functions where there is not uniqueness for initial values in a set with positive measure, has measure zero in all reasonable measures, e.g. supremumsnorm.

The assumption on f_i for $|u_i|$ large is satisfied if f_i increases or decreases in u_i faster than a linear function for $|u_i|$ large . This assumption is necessary in order to ensure a solution. Without this assumption there may not be a solution which satisfies the jump condition (1.4). If a physical problem is well-posed, then there is a bounded solution and the assumption on f_i for $|u_i|$ large is not relevant. The purpose with this assumption is only to ensure a solution also in the not well-posed problems.

The system (1.1) with initial value (1.2) is solved by solving component wise. The first equation is a scalar equation and existence and uniqueness theorems are well-known. This is stated as a separate theorem.

Theorem 2.2

The scalar Riemann problem

$$u_t + f(u)_x = 0$$

where f is locally Lipchitz continuous with inital value

$$u(x,0) = \begin{matrix} u_+ & \text{for } x > 0 \\ u_- & \text{for } x < 0 \end{matrix}$$

has a unique solution which may be described uniquely by a function u(s) where $s = \frac{x}{t}$. u(s) is piecewise continuous and there is a s_{min} and a s_{max} such that u(s) is constant for $s<s_{min}$ and $s>s_{max}$. In a discontinuity of u(s) there exists a unique entropy curve $w(\xi)$ such that

$$w'(\xi) = f(u(\xi)) - s\,u(\xi) - (f(u(s_\pm)) - s\,u(s_\pm)),$$

and $$w(\xi) \to u(s^\pm) \quad \text{when } \xi \to \pm\infty$$

In a discontinuity of u(s) the value on the left and right hand side of the discontinuity are denoted u_- and u_+ respectively.

The entropy curve is denoted as unique even though it is possible to shift the parameter.

Proof of this theorem is given in e.g. [10], [11] and [12].

The general problem is solved by induction on the number of equations. Assume that the problem is solved for n equations. We will then prove it for n+1 equations. The n+1 equation problem may be written as

$$v_t + g_x(u,v) = 0 \tag{2.1}$$

and

$$v(x,0) = \begin{matrix} v_+ & \text{for } x > 0 \\ v_- & \text{for } x < 0 \end{matrix} \tag{2.2}$$

u(s), $s=\frac{x}{t}$, is a known piecewise continuous function $u: R \to R^n$, which is constant for $s<s_{min}$ and $s>s_{max}$ for some s_{min} and s_{max}. Where u(s) is discontinuous, there exists an entropy curve

$$w'(\xi) = f(u(\xi)) - s\,u(\xi) - (f(u(s_\pm)) - s\,u(s_\pm)) \tag{2.3}$$

and $$w(\xi) \to u(s^\pm) \quad \text{when } \xi \to \pm\infty.$$

Similarly the solution v may be described by a function v(s) and for each discontinuity in v(s) there is an entropy curve $y(\xi)$.

The induction step in the proof of the main theorem is stated as a separate theorem.

Theorem 2.3

Assume g is continuous and that g_v is defined except in a finite number of points. Assume further that g_v is piecewise monotone and monotone in a finite number of intervals. Assume that there exists a v_{min} and a v_{max} such that

$$\text{either } g_v(u,v) < s_{min} \text{ or } g_v(u,v) > s_{max} \quad \text{for all } u, \text{ when } v < v_{min}$$

and

$$\text{either } g_v(u,v) < s_{min} \text{ or } g_v(u,v) > s_{max} \quad \text{for all } u, \text{ when } v > v_{max}.$$

Assume furthermore that $u(s):R \to R^n$, is piecewise continuous and constant for $s < s_{min}$ and $s > s_{max}$ and where $u(s)$ is discontinuous there exists an entropy curve $w(\xi)$. Then there exists a unique solution to the Riemann problem

$$v_t + g_x(u,v) = 0$$

$$v(x,0) = \begin{matrix} v_L & \text{for } x < 0 \\ v_R & \text{for } x > 0. \end{matrix}$$

There also exists an integral curves for each shock in v. These integral curves are unique except for g(u,v) in a set with measure zero and for a set with measure zero in the (v_L,v_R) plane which depends on the function g(u,v).

If the entropy curves $w(\xi)$ are not unique, the solution v(x,t) is not always unique.

In the argument below we assume that there is a fixed left value v_L for v.

The values of v_R which can be connected to a given v_L in the sector $\frac{x}{t} < s$ is found. When this maximum speed is large enough, it is possible to connect the fixed v_L to all possible v_R values. In describing the possible v_R values which may be connected to v_L, the function $h_s(v)$ is used in addition to the function g(u(s),v).

Definition of the function $h_s(v)$

$$h_s(v) = \begin{cases} g(u(s),v) \text{ if it is possible to connect } v \text{ to } v_L \text{ with speed } s \\ \text{else} \\ \text{a linear function with slope } s \text{ which makes } h_s(v) \text{ continuous to the left (and for } v \text{ large to the right) of intervals where it is possible to connect } v \text{ to } v_L. \end{cases}$$

It is denoted that it is possible to connect v_R to v_L with speed equal s iff there is an entropy solution in the sector $\frac{x}{t} < s$ which is equal to v_L for $\frac{x}{t} < s_-$ for some $s_- < s$, and equal to v_R for $\frac{x}{t} = s$.

We will prove that the function $h_s(v)$ has the following properties:

<u>Properties to $h_s(v)$</u>

- $h_s(v)$ is continuous and $h_s'(v) < s$ where defined.
- $h_s(v) = g(u(s),v)$ in a finite number of intervals. An interval may consist of one point. There is at least one interval.
- Between these intervals $h_s(v)$ is linear with slope s.
- There exist a v_0 such that for $v > v_0$ we have

  either $\quad h_s(v) < g(u(s),v)$,

  or $\quad h_s(v) < g(u(s),v)$.
- There exist a v_1 such that for $v < v_1$ we have

  either $\quad h_s(v) = g(u(s),v)$,

  or $\quad h_s(v) > g(u(s),v)$.

See figure 2.1 for a typical $h_s(v)$ and $g(u(s),v)$.

We may then start with the proofs.

<u>Proposition 2.4</u>

Assume g and u satisfy the assumptions in Theorem 2.3. Then the function $h_s(v)$ has the properties listed and the v(s) function which connects v_L with v_R is always unique. The entropy curves $w(\xi)$ are unique except for g(u,v) in a set with measure zero and for a finite number of v_R which depend on g(u,v).

Before this proposition is proved, some lemmas must be proved.

<u>Lemma 2.5</u>

Assume g is continuous. Then Proposition 2.4 is valid for $s<s_-$ if $u(s) = u_-$ for $s<s_-$.

<u>Proof of lemma 2.5.</u>

When u(s) is constant, the system (2.1) and (2.2) is equivalent to the scalar problem. The solution is then well-known. If v_L is smaller than v_R, the solution is described by the convex envelope from v_L to v_R, and if v_L is larger than v_R, the solution is described by the concave envelope from v_L to v_R. It is easily seen that Proposition 2.4 is satisfied. See figure 2.2 for a typical $h_s(v)$ when u(s) is constant. The function v(s) and the entropy curves are always unique.●

Lemma 2.6

Assume g is continuous, that Proposition 2.4 is valid for $s=s_0$ and that $u(s)$ is continuous for $s\in[s_0,s_1]$. Then Proposition 2.4 is valid for $s=s_1$.

Proof of lemma 2.6.

When $u(s)$ is continuous, we will prove that the solution of (2.1) - (2.3) is a combination of smooth rarefaction waves and shocks in the v variable.

Let v_0 be an arbitrary point where

$$h_{s_0}(v_0) = g(u(s_0),v_0).$$

Assume first that $g_v(u(s_0),v_0) \neq s_0$.

The equation may be rewritten to

$$- s\, v_s + g_u(u,v)\, u_s + g_v(u,v)\, v_s = 0.$$

Therefore there is a rarefaction wave starting in v_0 defined by

(2.4) $\qquad v(s_0) = v_0$

(2.5) $\qquad v_s(s) = \dfrac{g_u\, u_s}{s - g_v}$.

These curves are well-defined as long as $g_v(u(s),v(s))\neq s$.

Two curves cannot cross each other, i.e. if $v_1(s_1) < v_2(s_1)$, then $v_1(s) < v_2(s)$ for all s.

In (v,g) plane the curves $(v(s),g(u(s),v(s)))$ are parallel with slope s, i.e.

$$\frac{g_s}{v_s} = \frac{g_u\, u_s + g_v\, v_s}{v_s} = \frac{v_s(s-g_v) + g_v v_s}{v_s} = s.$$

Secondly, assume that $g_v(u(s),v(s)) = s$ either for $s = s_0$ or for $s > s_0$, then there is a shock in the v variable. Since $u(s)$ is continuous, this shock is exactly as a shock in the scalar equation, i.e. we may connect a value v_+ to the right to a v_- value with speed

$$s = \frac{g(u(s),v_-) - g(u(s),v_+)}{v_- - v_+}.$$

If $g(u(s),v) > v_- + s\,(v - v_-)$ for v between v_- and v_+, then $v_+ > v_-$ and if $g(u(s),v) < v_- + s\,(v - v_-)$ for v between v_- and v_+, then $v_- > v_+$.

In these shocks the entropy curves are unique exactly as in the scalar equation. Also these curves have slope s in the (v,g) plane, therefore they do not cross the rarefaction curves.

Thus for every point v_0 where $h_{s_0}(v_0) = g(u(s_0),v_0)$, there starts a rarefaction or a combined rarefaction and shock curve in the v variable. When s increases from s_0 to s_1, these curves defines the function $h_s(v)$. Especially $h_{s_1}(v)$ is well-defined. It is easily seen that $h_{s_1}(v)$ satisfies the properties of $h_s(v)$ listed.

The construction in this proof shows that there is always a unique solution v(s) with unique entropy curves $w(\xi)$ for $s_0 < s < s_1$. If there is a v_R value for $s=s_0$ which has multiple entropy curves, this leads to multiple entopy curves for v_R' which is the value v_R is transformed to using the construction in this proof.●

Thus we are left with the most difficult case where there is a shock in u. Assume u(s) is discontinuous in s_0 and a singel shock connects the left and right values u_- and u_+ respectively. Assume further that Proposition 2.4 is valid for s_0-, and that there exists a piecewise monotone entropy curve $w(\xi)$ such that

$$w(\xi) \to u_\pm \qquad \text{when} \qquad t \to \pm \infty.$$

We write $h_-(v)$ and $h_+(v)$ instead of $h_{s_0^-}(v)$ and $h_{s_0^+}(v)$ respectively.

We will first prove that there are integral curves starting and ending from almost everywhere on $h_-(v)$.

<u>Lemma 2.7</u>

Asssume g and u satisfy the assumptions in Theorem 2.3.
Consider the integral curves

$$v_{b,c}(0) = b, \qquad v_{b,c}'(\xi) = g(w(\xi),v_{b,c}(\xi)) - s_0 v(\xi) - c$$

where b and c are constants. Then

1. If $b_1 < b_2$, then $v_{b_1,c}(\xi) < v_{b_2,c}(\xi)$ for all ξ.

Consider convergence when ξ decreases to $-\infty$:

2. For all b values $v_{b,c}(\xi)$ converges to a v_- where $g(u_-,v_-) = c + s_0 v_-$ or diverges to ∞ or $-\infty$ when ξ decreases to $-\infty$.
3. For every v_- value where $g(u,v) - s_0 v$ increases in v in a neighbourhood to v_-, there exist d and e such that for $d < b < e$ $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$.
4. For every v_- value where $g(u,v) - s_0 v$ decreases in v in a neighbourhood to v_-, there exists a unique b such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$.

When ξ increases to ∞, 2., 3. and 4. is stil valid but there is uniqueness when $g(u,v) - s_0 v$ increases and convergence for b in an interval when $g(u,v) - s_0 v$ decreases.

Proof of Lemma 2.7

For siplicity we assume $s_0 = 0$.

1. It is trivial to prove 1.

We will only prove the lemma when ξ decreases to $-\infty$.

2. A necessary condition for convergence is that $v'_{b,c}(\xi)$ vanishes when ξ decreases to $-\infty$. This is only satisfied for v_- such that $g(u_-,v_-) = c$. It is then trivial to prove 2.

Assume $g(u_-,v_-) = c$ and $g(u_-,v)$ is monotone decreasing/increasing in a neigbourhood to v_-. Then according to the assumptions on $g(u,v)$, there exist a ξ_0 such that for $\xi < \xi_0$, $g(w(\xi),v)$ is monotone decreasing/increasing in an interval (d,e) with $d < v_- < e$. Therefore there exists a unique function $a(\xi)$ such that $g(w(\xi),a(\xi)) = c$ for $\xi < \xi_0$. $a(\xi) \to v_-$, when ξ decreases to $-\infty$.

3. Assume $g(u,v)$ is increasing in a neigbourhood to v_-. Then the point $a(\xi)$ is attractive; $v(\xi)$ is moving towards $a(\xi)$ when $\xi < \xi_0$. Therefore $v_{b,c}(\xi)$ converges to v_- when $d < v_{b,c}(\xi_0) < e$. This interval is transfered to another interval when ξ_0 is replaced by 0.

4. Assume $g(u,v)$ is decreasing in a neigbourhood to v_-. Then the point $a(\xi)$ is repulsive; $v(\xi)$ is always moving away from $a(\xi)$. According to 2., $v_{b,c}(\xi)$ is always converging or diverging to ∞ or $-\infty$. We will prove that $v_{b,c}(\xi)$ always converges or diverges to the left/right of v_- for b in an open interval. Let us first prove that there is an open interval to the left.

For general initial value problems we have

$$\lim_{b \to b_0} v_{b,c}(\xi) = v_{b_0,c}(\xi) \qquad \text{for all } \xi.$$

Assume $v_{b,c}(\xi)$ converges to the left of v_-. Then $v_b(\xi) < d - \delta$ for $\xi < \xi_1$ for some ξ_1. Then also $v_{b+\epsilon,c}(\xi) < d - \frac{\delta}{2}$ for ϵ small and therefore also $v_{b+\epsilon,c}(\xi)$ converges to the left of v_-. Then the interval for convergence to the left of v_- is open. The proof for convergence to the right of v_- is correspondingly. Therefore there exists at least one point between these two open intervals where $v_{b,c}(\xi)$ converges to v_-.

Assume $v_{b_1,c}(\xi)$ and $v_{b_2,c}(\xi)$ both converges to v_-. It is easily seen that $| v_{b_1,c}(\xi) - v_{b_1,c}(\xi) |$ increases when ξ decreases, so there must be a unique value of b such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$.●

Figure 2.3a shows a typical function $g(\underline{u},v)$ and a c value. Figure 2.3b shows where $v_{b,c}(\xi)$ converges when ξ decreases to $-\infty$ depending on the b value.

Lemma 2.8

Let $v_i(\xi)$ i=1,2, be two integeral curves satisfying

$v_i'(\xi) = g(w(\xi),v_i(\xi)) - s_0v_i(\xi) - c_i$,

$v_i(\xi)$ converges to $v_{i,-}$ when ξ decreases to $-\infty$ and

$h_-(v_{i,-}) = g(u_-,v_{i,-})$ for i=1,2 and

$v_{1,-} < v_{2,-}$.

Then $v_1(\xi) < v_2(\xi)$ for all ξ.

Proof of lemma 2.8

For simplicity we assume $s_0 = 0$.

$v_i'(\xi)$ vanishes when ξ decreases to $-\infty$. Therefore

$$c_i = g(u_-,v_{i,-}) = h_-(v_{i,-}).$$

According to the assumptions on $h_-(v)$ and since $v_{1,-} < v_{2,-}$, we have $c_1 > c_2$.

Assume the lemma is not correct. Let ξ_0 be the smallest value of ξ such that $v_1(\xi_0) = v_2(\xi_0)$. Then $v_1(\xi_0-\epsilon) < v_2(\xi_0-\epsilon)$. But

$$v_1'(\xi_0) = g(w(\xi),v_1(\xi_0)) - c_1 < g(w(\xi),v_2(\xi_0)) - c_2 = v_2'(\xi)$$

and therefore $v_1(\xi_0-\epsilon) > v_2(\xi_0-\epsilon)$ for ϵ small and positive. This is a contradiction and therefore $v_1(\xi) < v_2(\xi)$ for all ξ.●

Then we can consider the case where u(s) is discontinuous, i.e. there is a shock in one of the equations higher up in the system of equations.

Lemma 2.9

Assume g and u satisfy the assumtions in Theorem 2.3. Assume further that u(s) is discontinuous for $s=s_0$ and that Proposition 2.4 is valid for s_0-. Then Proposition 2.4 is valid for s_0+.

Proof of lemma 2.9.

For simplicity we assume $s_0 = 0$.

In Lemma 2.7 we defined the integral curves

$$v_{b,c}(0) = b \text{ and } v_{b,c}'(\xi) = g(w(\xi),v_{b,c}(\xi)) - c.$$

$v_{b,c}$ converges to v_- or diverges to ∞ or $-\infty$ when ξ decreases to $-\infty$. In the lemma we proved that if $g(u,v)$ is monotone decreasing in v in a neighbourhood to v_- then there is a unique (b,c) value such that $v_{b,c}(\xi)$ converges to v_- when ξ decreases to $-\infty$. Lemma 2.8 shows that two integral curves which converges to points on $h_-(v)$ does not pass each other. Then we may define the function $\gamma(b)=c$ if $v_{b,c}(\xi)$ converges to v_- and $h_-(v_-)=c$. It is easy to see that $\gamma(b)$ is well-defined and continuous. See figure 2.4. $\gamma(b)$ is a monotone decreasing function of b.

In a similar way the integral curves $v_{b,c}(\xi)$ converges to a v_+ or diverges to ∞ or $-\infty$ when ξ increases to ∞. Let us study the (b,c) values where the curve $v_{b,c}(\xi)$ converges to v_+ when ξ increases to ∞. For convergence to v_+ the situation is changed; there is a single (b,c) value for which $v_{b,c}(t)$ converges to a point where $g(u_+,v)$ is monotone increasing in a neighbourhood of v_+ and an interval with b values for which $v_{b,c}(\xi)$ converges to a v_+ where $g(u_+,v)$ is monotone decreasing in a neighbourhood of v_+. See figure 2.5 where the different (b,c) values where $v_{b,c}(\xi)$ converges to v_+ and $g(u_+,v)$ is monotone increasing in a neighbourhood of v_+ is shown.

Since $\gamma(b) = c$ is continuous, it crosses the curves in the (b,c) plane where the corresponding integral curve $v_{b,c}(\xi)$ converges to to v_+ and $g(u_+,v)$ is monotone increasing in a neighbourhood of v_+.

The definition of $h_s(v)$ is in this situation that $h_+(v) = c$ if $\gamma(b) = c$ and $v_{b,c}(\xi)$ converges to v when ξ increases to ∞. Since $\gamma(b)$ is continuous $h_+(v)$ is well-defined for all v. It is trivial to see that $h_+(v)$ satisfies the listed properties of $h_s(v)$. Figure 2.6 shows typical $h_-(v)$ and $h_+(v)$.

There is usually a unique entropy curve $y(\xi)$ from v_- to v_+ except when $g(u_-,v)$ is monotone increasing in a neighbourhood of v_- and $g(u_+,v)$ is monotone decreasing in a neighbourhood of v_+. In this situation there is usually a continuum of entropy curves connecting v_- and v_+. Since there is only a finite number of v_- such that $h_-(v_-) = g(u_-,v_-)$ and $g(u_-,v)$ is monotone increasing in a neighbourhood of v_-, there is only a finite number of v_+ values where there are not a unique connection from v_-.

A finite number of v_+ values are passed through for v_R in an interval. See figure 2.7 for an example. In figure 2.7a $g(u(0),.)$ and $h_0(.)$ is shown. $u(s)$ is constant for $s>0$. Then we see that $v(0+) = d$ for $b < v_R < e$. Thus a jump from $v_-= a$ to $v_+= d$ is passed through for

for v_R in an interval. Figure 2.7b shows v(s) for v_R = d and figure 2.7c shows v(s) for v_R = c. In both figures there is a jump from a to d with speed 0. If the connection from v_L to v_- = c is not unique, then there is not uniqueness for v_R in an interval. It is therefore essential that not any of the finite number of v_- values which do not have a unique connnection to v_L, is connected to the finite number of v_+ values which are passed through for v_R in an interval. It is easily seen that this only happens for g(u,v) in a set with measure zero.●

In chapther 3 an example shows a flux function where there is a continuum of solutions for (u_-,u_+) in a set with positive measure.

We may then prove the proposition.

Proof of Proposition 2.4

u(s) is piecewise continuous and constant for s small and s large. From Lemma 2.5 it follows that the proposition is valid for s small. Lemma 2.6 implies that if u(s) is continuous in an interval and Proposition 2.4 is correct at the left end of the interval then it is correct at the right end of the interval. Furthermore Lemma 2.9 implies that if u(s) is discontinuous and Proposition 2.4 is correct to the left of the discontinuity, then it is correct to the right of the discontinuity. Since u(s) only is dicontinuous in a finite number of points, the finite number of v values where there are a continuum of entropy curves is kept finite. Therefore the proposition is valid for any s.●

In order to prove existence for every initial value the following obvious lemma is needed.

Lemma 2.10

Assume g and u satisfy the assumptions in Theorem 2.3. Then for each v value there exists a s_+ such that for $s > s_+$, $h_s(v) = g(u_+,v)$.

Proof of Theorem 2.3.

The theorem follows easily from Proposition 2.4 and Lemma 2.10.●

We may then prove Theorem 2.1.

Proof of Theorem 2.1.

The theorem is proved by induction. For n=1 the theorem is the well-known result stated in Theorem 2.2. Theorem 2.9 is used as the induction step.

For n=2 there may be several entropy curves, but the solution is still unique. For n>2 this may lead to several solutions. If (u_-,u_+) is in a set with measure zero, then also the composed (u_-,v_-,u_+, v_+) is in a set with measure zero. Similarly if f is in a set with measure zero, then also the composed function (f,f_{n+1}) is in a set with measure zero. Therefore the solution is unique except for the flux function in a set with measure zero and for the initial value in a set with measure zero.●

3 SOME CHARACTERISTICA OF THE SOLUTION

In this chapter we study some of the characteristica of the solution of lower triangular hyperbolic systems. First we show that the Lax entropy inequalities are not always satisfied. Furthermore we prove that the solution does not depend continuously on the data. An example shows a flux function where there is a continuum of solutions for the initial values in a set with positive measure.

For genuinly nonlinear and strictly hyperbolic systems the following inequalities

$$\lambda_k(u_+) < s < \lambda_{k+1}(u_+)$$

and

$$\lambda_{k-1}(u_-) < s < \lambda_k(u_-)$$

where λ_k are the ordered eigenvalues to the system, where proved by Lax [8] for local solutions. In lower triangular hyperbolic systems the eigenvalues equals $\lambda^i = \dfrac{\partial f_i}{\partial u_i}$. Notice that the superscript does not indicate the order of the eigenvalue. Assume that there is a simple rarefaction solution in equations 1,...,k-1. Then there is a shock with speed s in equation k. This shock influences the solution in equations k+1,...,n. Thus λ^i is larger or smaller than s on both side of the shock for i=1,...,k-1. For i=k the eigenvalues appear as in the scalar equation i.e. $\lambda^k(u_+) < s < \lambda^k(u_-)$. According to the proof of lemma 2.9 it is easily seen that for i > k $\lambda^i(u_-)$, $\lambda^i(u_+) < s$ or $\lambda^i(u_-)$, $\lambda^i(u_+) > s$ for local solutions. This is according to the Lax entropy inequalties. However for non-local solutions we may have $\lambda^i(u_-) < s < \lambda^i(u_+)$ or $\lambda^i(u_+) < s < \lambda^i(u_-)$. $\lambda^i(u_+) < s < \lambda^i(u_-)$ corresponds to the situation where the solution is not unique. In the second part this chapter we show this in an example. It is also easy to find examples with n=2 where $\lambda^i(u_-) < s < \lambda^i(u_+)$. We conclude that the Lax shock inequalites are not correct for non-local solutions of non-stricktly hyperbolic systems. See also Johansen and Winther [5].

The solution in the scalar equation depend continuously on the data, see Lucier [8] and Holden, Holden and Høegh-Krohn [3]. For the scalar equation the following theorem is valid.

Theorem 3.1

If f and g are Lipschitz continuous functions, u_0 and $v_0 \in BV(R)$ and u and v are the solutions of

$$\begin{aligned} u_t + f(u)_x &= 0 && \text{for } x\in R \text{ and } t>0 \\ u(x,0) &= u_0(x) && \text{for } x\in R \end{aligned}$$

and

$$\begin{aligned} v_t + g(v)_x &= 0 && \text{for } x\in R \text{ and } t>0 \\ v(x,0) &= v_0(x) && \text{for } x\in R, \end{aligned}$$

then for any $t>0$

$$||u(.,t)-v(.,t)||_{L_1} < ||u_0(x)-v_0(x)||_{L_1} + ||f-g||_{Lip} \min(|u_0|_{BV(R)}, |v_0|_{BV(R)}),$$

where we have defined

$$||g||_{Lip} = \sup_{x\neq y} \left| \frac{g(x)-g(y)}{x-y} \right| .$$

In lower triangular hyperbolic systems the solution does not depend continuously on the data. This is connected to the nonuniqueness of the solution. In the following example we approach a point where the solution is not unique along different curves where the solution is unique.

In the example $n=3$. We consider one equation at a time.

$$f_1(u_1) = - u_1^2$$

and

$$u_1(x,0) = \begin{matrix} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{matrix}$$

The solution is easily found to be

$$u_1(x,t) = \begin{matrix} -1 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{matrix}$$

See figure 3.1. The definition of f_2 is more complicated, namely

$$f_2(u_1,u_2) = \begin{matrix} g_1(u_2) & \text{for } u_1 < -1 \\ \frac{1}{2}(1-u_1)\, g_1(u_2) + \frac{1}{2}(1+u_1)\, g_2(u_2) & \text{for } -1 < u_1 < 1 \\ g_2(u_2) & \text{for } 1 < u_1 \end{matrix}$$

where

$$g_1(u) = \begin{matrix} |u| & \text{for } u < 1 \\ 2-u & \text{for } u > 1 \end{matrix}$$

and

$$g_2(u) = -2-u.$$

See figure 3.2 for the definition of f_2. We use two different initial values in the Riemann problem. The initial values are

$$u_2^{\pm}(x,0) = \begin{matrix} -1 & \text{for } x < 0 \\ -2+\epsilon & \text{for } x < 0 \end{matrix}$$

respectively

$$u_2^-(x,0) = \begin{cases} -1 & \text{for } x < 0 \\ -2-\epsilon & \text{for } x < 0. \end{cases}$$

for $\epsilon > 0$. The exact solutions are

$$u_2^+(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < -\delta \\ 2+\epsilon & \text{for } -\delta < \frac{x}{t} < 0 \\ -2+\epsilon & \text{for } 0 < \frac{x}{t} \end{cases}$$

and

$$u_2^-(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 0 \\ -2+\epsilon & \text{for } 0 < \frac{x}{t} \end{cases}$$

for $\delta > 0$. δ depends on ϵ, and δ vanishes when ϵ vanishes.See figure 3.3 and 3.4. We see that when the right hand value approaches -2 then these two solutions become identical. But the entropy curves with speed 0 do not converge. This becomes more evident when we add the third equation

$$f_3(u_2,u_3) = \begin{cases} g_3(u_3) & \text{for } u_2 < 0 \\ \frac{1}{2}(2-u_2)\, g_3(u_3) + u_2 g_4(u_3) & \text{for } 0 < u_2 < 2 \\ g_4(u_3) & \text{for } 2 < u_2 \end{cases}$$

where

$$g_3(u) = |u|$$

and

$$g_4(u) = |u| + 2.$$

See figure 3.5. The initial value is

$$u_3(x,0) = \begin{cases} -1 & \text{for } x < 0 \\ 1 & \text{for } x < 0. \end{cases}$$

The solution depends on the initial value for u_2.

$$u_3^+(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ -2-\epsilon & \text{for } -1 < \frac{x}{t} < -\delta \\ 0 & \text{for } -\delta < \frac{x}{t} < 0 \\ 2 & \text{for } 0 < \frac{x}{t} < 1 \\ 1 & \text{for } 1 < \frac{x}{t} \end{cases}$$

and

$$u_3^-(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 1 \\ 1 & \text{for } 1 < \frac{x}{t} \end{cases}$$

See figure 3.6 and figure 3.7. When the right hand initial value for u_2 equals -2, there is a continuum with entropy curves between the two entropy curves we find when the inital value approaches -2 from both sides. The corresponding solution for u_3 is changing from u_3^+ to u_3^-. The sector with value 0 is increasing and finally ends up as in u_3^-.

We will then give an example with a flux function which gives a continuum with solutions for the initial values in a set with positive measure. The flux function and the initial data is only a minor modification of the previous example.

$$f_1^*(u_1) = \begin{cases} -u_1^2 & \text{for } x < 1 \\ 2u_1 - 3 & \text{for } x > 1 \end{cases}$$

and $$u_1(x,0) = \begin{cases} -1 & \text{for } x < 0 \\ 2 & \text{for } x > 0. \end{cases}$$

The solution is easily found to be

$$u_1^*(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < 0 \\ 1 & \text{for } 0 < \frac{x}{t} < 2 \\ 2 & \text{for } 2 < \frac{x}{t}. \end{cases}$$

See figure 3.9. The definition of f_2^* is

$$f_2^*(u_1,u_2) = \begin{cases} f_2(u_1,u_2) & \text{for } u_1 < 1 \\ (2-u_1)\, f_2(u_1,u_2) + (u_1-1)\, g_5(u_2) & \text{for } 1 < u_1 < 2 \\ g_5(u_2) & \text{for } 2 < u_1 \end{cases}$$

where

$$g_5(u) = \begin{cases} 3\,|u+3| - 3 & \text{for } u < -1 \\ 2 - u & \text{for } u > -1. \end{cases}$$

See figure 3.10 for the definition of f_2^*. The initial value in this equation is

$$u_{2,a}^*(x,0) = \begin{cases} -1 & \text{for } x < 0 \\ a & \text{for } x < 0 \end{cases}$$

where $-3.2 < a < -.75$. The solution is then

$$u^*_{2,a}(x,t) = \begin{cases} -1 & \text{for } \frac{x}{t} < -1 \\ 0 & \text{for } -1 < \frac{x}{t} < 0 \\ -2 & \text{for } 0 < \frac{x}{t} < s(a) \\ a & \text{for } s(a) < \frac{x}{t} \end{cases}$$

where $s(a) = \dfrac{f^*_2(2,-2) - f^*_2(2,a)}{-2 - a}$. We see that $2 < s(a) < 3$. In the solution of $u^*_2(x,t)$ there is a continuum of entropy curves in the jump with speed 0 exactly as in the previous example. There is also a jump in $u^*_2(x,t)$ with speed 2, but $u^*_2(x,t)$ is equal -2 on both sides. This jump is essential since $\dfrac{\partial f^*_2}{\partial u_2}$ is smaller than the speed of the jump before the jump and larger after the jump. In the jump with speed 2 the Lax entropy inequalities are not satisfied.

In the third equation we chose $f^*_3(u_2,u_3) = f_3(u_2,u_3)$ and $u^*_3(x,0) = u_3(x,0)$. The solutions are $u^+_3(x,t)$, $u^-_3(x,t)$ both with $\epsilon=0$, and a continuous spectrum between these two solutions. We may pertube the initial values and still get a continuum of solutions. The flux function may not be pertubed since it is essential that $f^*_2(u_1,u_2) = 0$ for $1 < u_1 < 2$ and $u_2 = -2$. The example shows that there exists flux functions such that there is a continuum of solutions for the initial values in a set with positive measure.

4 NUMERICAL METHODS FOR LOWER TRIANGULAR HYPERBOLIC SYSTEMS

There are several different numerical methods for the scalar equation. It is possible to generalize most of these to lower triangular hyperbolic systems. Here we will use a method which follow the proofs in chapter 2, except that the entropy curves are found by a numerical method for the integral curve.

For special flux functions it may be easy to find the $h_s(v)$ functions. For example for g convex in the v variable there are v_1 and v_2 values depending on s, such that

$$\begin{aligned} h_s(v) &= g(u(s),v) && \text{for } v < v_1, \\ h_s(v) &> g(u(s),v) && \text{for } v_1 < v < v_2 \text{ and} \\ h_s(v) &< g(u(s),v) && \text{for } v_2 < v. \end{aligned}$$

For general flux functions it is cumbersome to handle the whole $h_s(v)$ function. Instead a shooting method is valuable. A shooting method runs as follows:

Try to connect the v_L value to any v_R value. This is done by following the curve $u(s)$. When $u(s)$ is constant, convex or concave envelopes are used. The integral curves (2.4) and (2.5) are used when $u(s)$ is continuous but not constant. Use an ordinary numerical method for (2.4) and (2.5). It is more difficult when $u(s)$ is discontinuous since there is no initial value for the integral curve. Numerically, this is solved by setting $v(\xi_0) = g(w(\xi_0),v_-)$ for ξ_0 small. Following the $u(s)$ curve we finally reach a v_R which probably is different from the v which was wanted. This scheme is monotone, i.e. if we move a little shorter in v the variable for a specific value of s, then the v_R which is found is smaller than the original v_R independently of what happens for larger s values. Thus it is easy to approximate any v_R.

If we assume that the eigenvalues of (1.1) are in distinct intervals, when u varies in the phase space, it is easy to find the solutions for shocks in $u(s)$. In this case it is not necessary to use the entropy curves since the shocks are uniquely defined by the equation

$$s = \frac{g(u_+,v_+) - g(u_-,v_-)}{v_+ - v_-}. \tag{4.1}$$

If f_i, $i=1,2,\ldots,n$ are approximated by piecewise linear functions the solution only consists of shocks and therefore is piecwise constant.

Thus if f_i, $i=1,2,\ldots,n$ are all piecewise linear and the eigenvalues are in distinct intervals there is no need to use any integral curves. Hence it is possible to solve the problem exactly using only convex and concave envelopes and shocks with speed defined by (4.1). See [3] for a similar technique for the scalar equation

Acknowledgement

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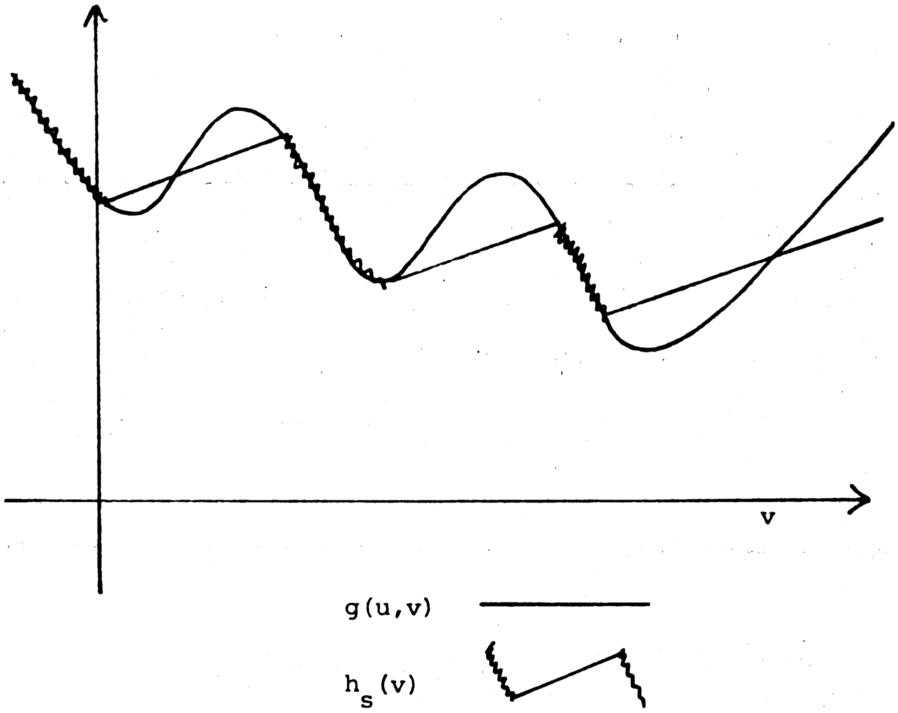

Figure 2.1 A typical $g(u,v)$ and $h_s(v)$, $s>0$.

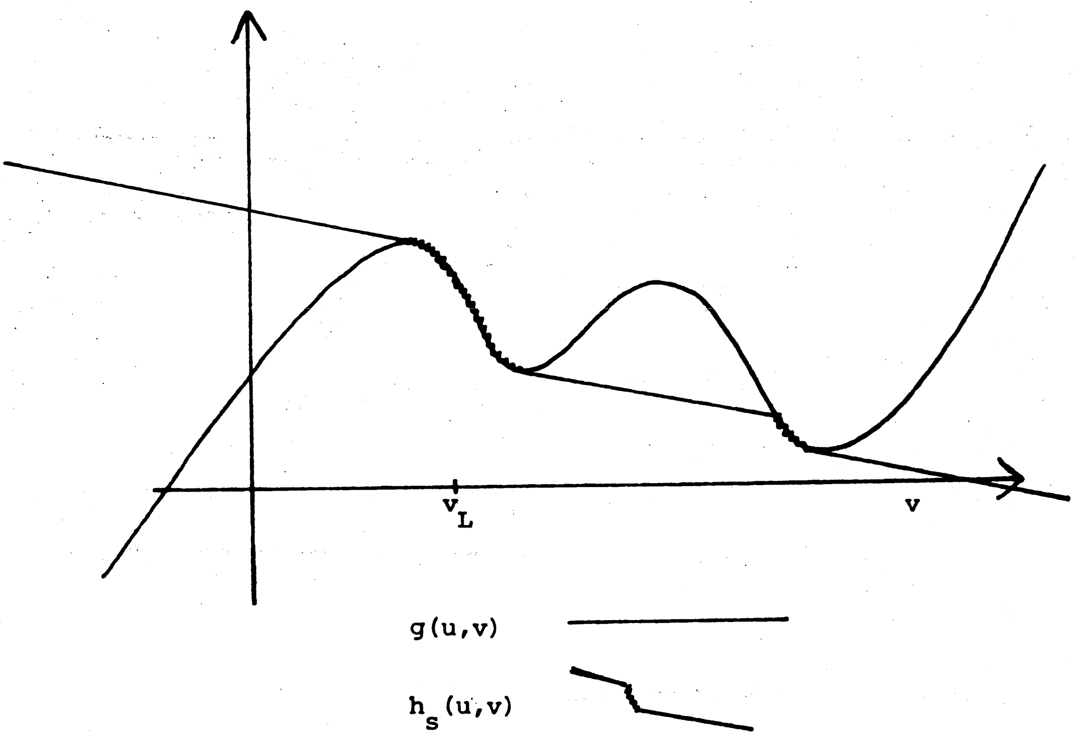

Figure 2.2 $g(u,v)$ and $h_s(v)$ for $u(s)$ constant, $s<0$

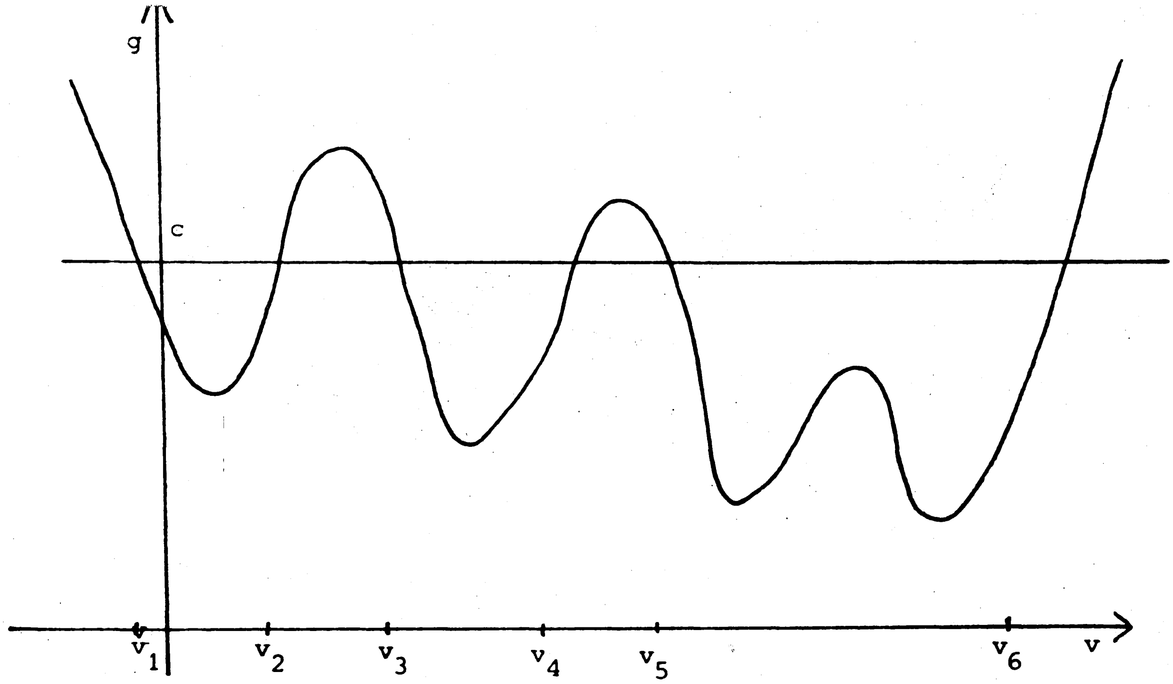

Figure 2.3a $g(u_-,v)$ and the constant c.

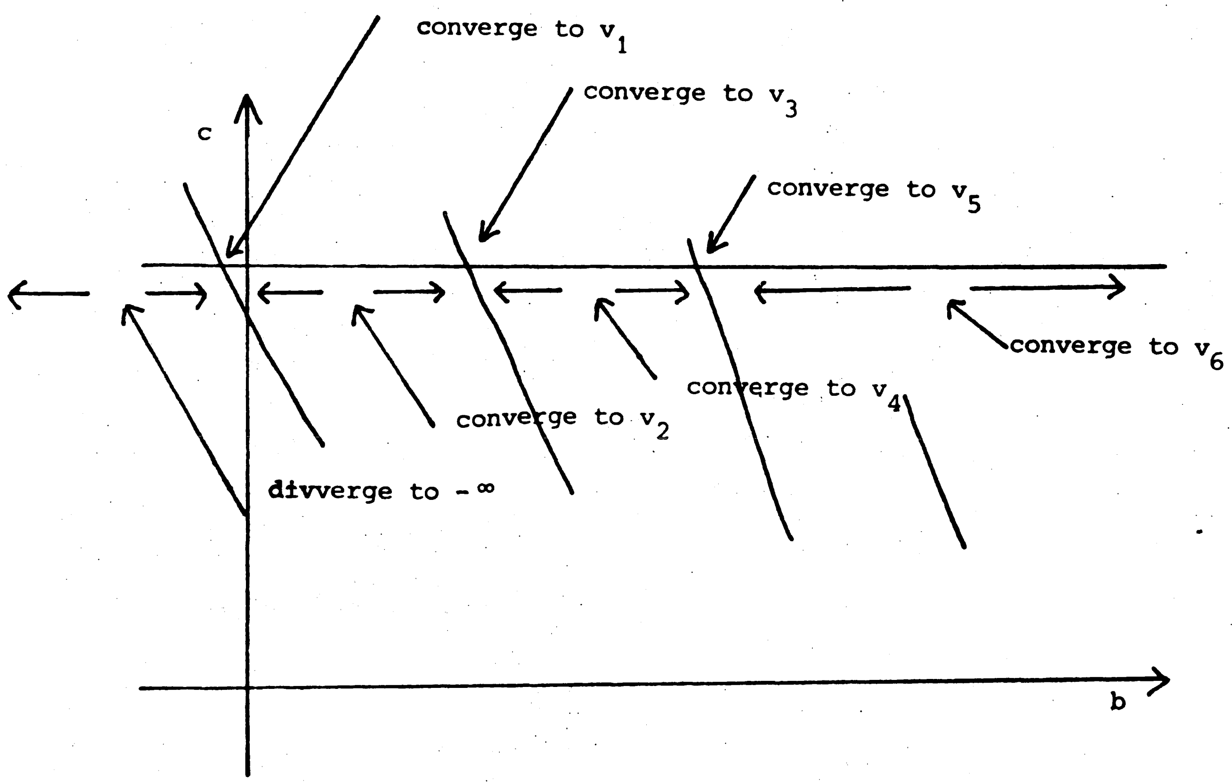

Figure 2.3b Convergence when $\xi \to -\infty$

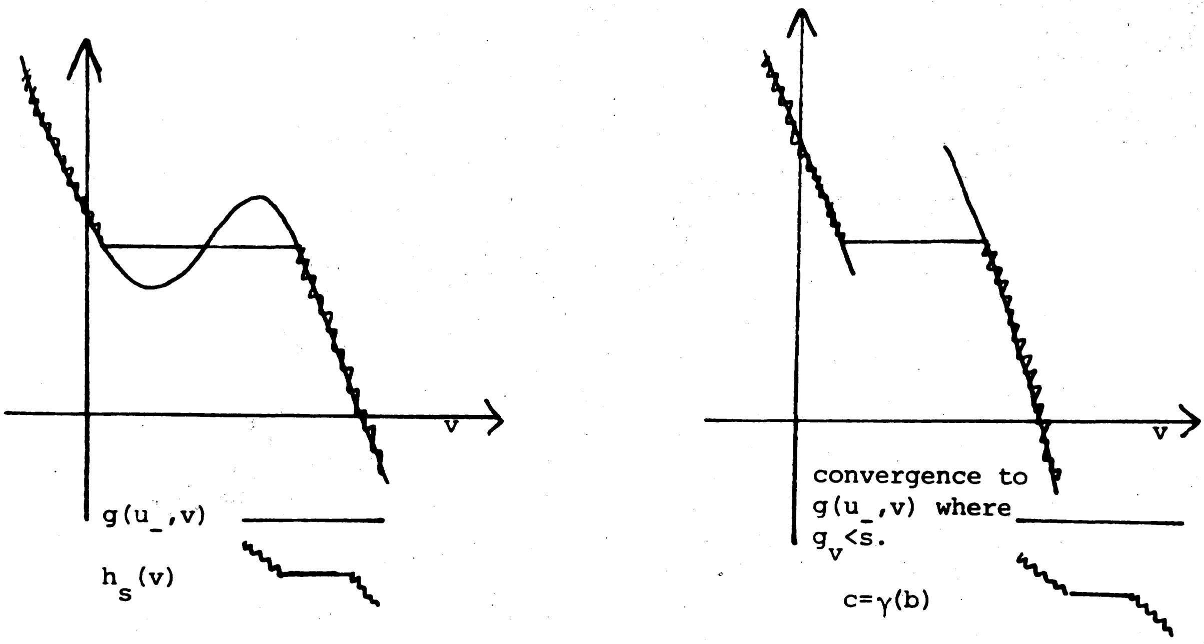

Figure 2.4 $h_s(v)$ and $\gamma(b)$. $s=0$ in figure.

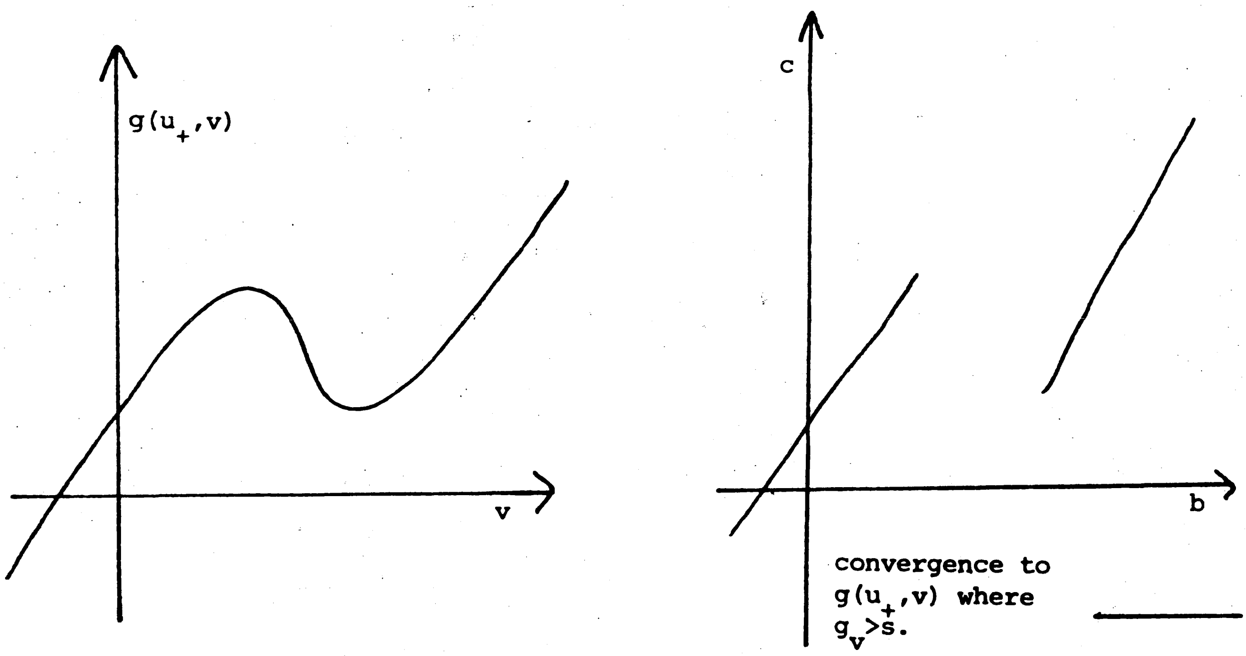

Figure 2.5 Convergence to $g(u_+,v)$

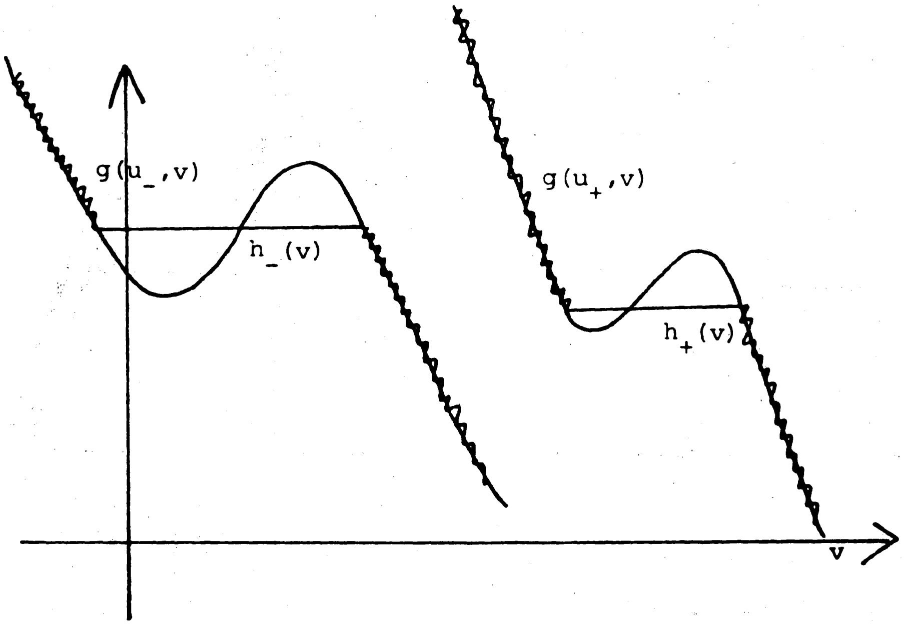

Figure 2.6 $h_-(v)$ and $h_+(v)$

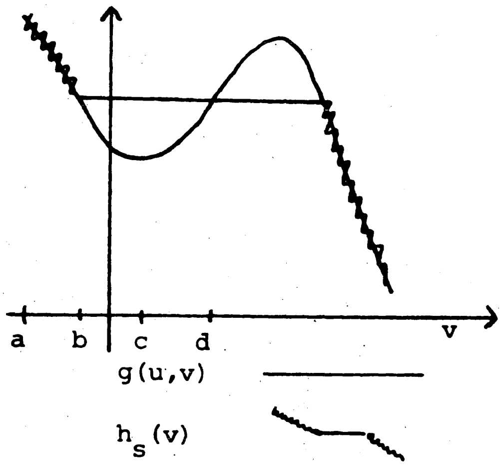

Figure 2.7a

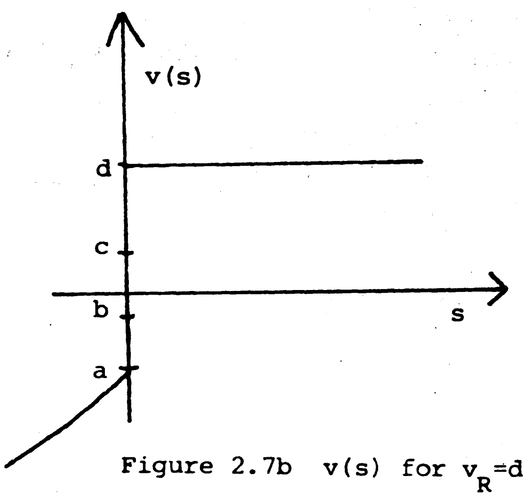

Figure 2.7b v(s) for v_R=d

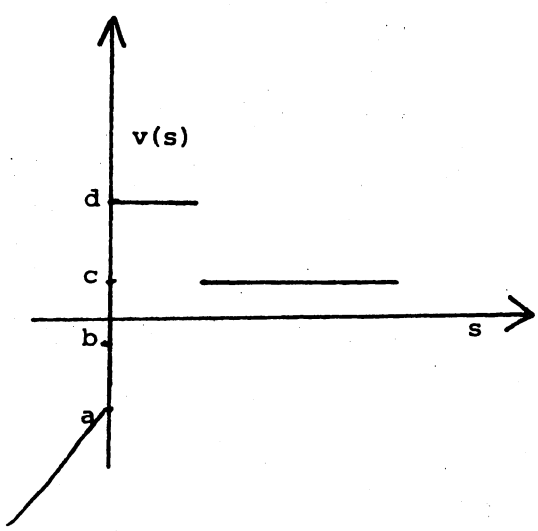

Figure 2.7c v(s) for v_R=c

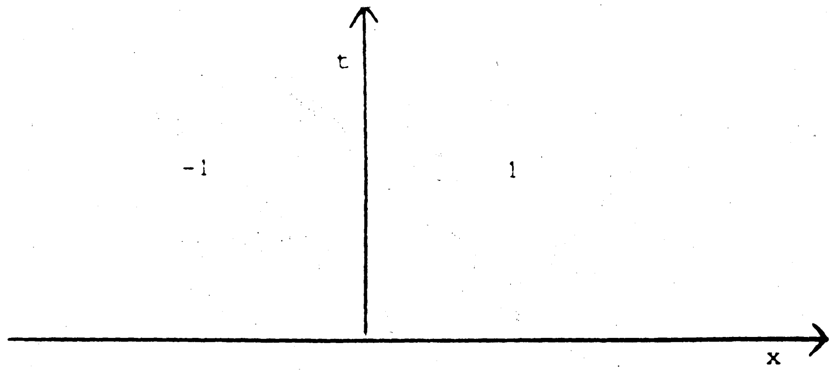

Figure 3.1. $u_1(x,t)$

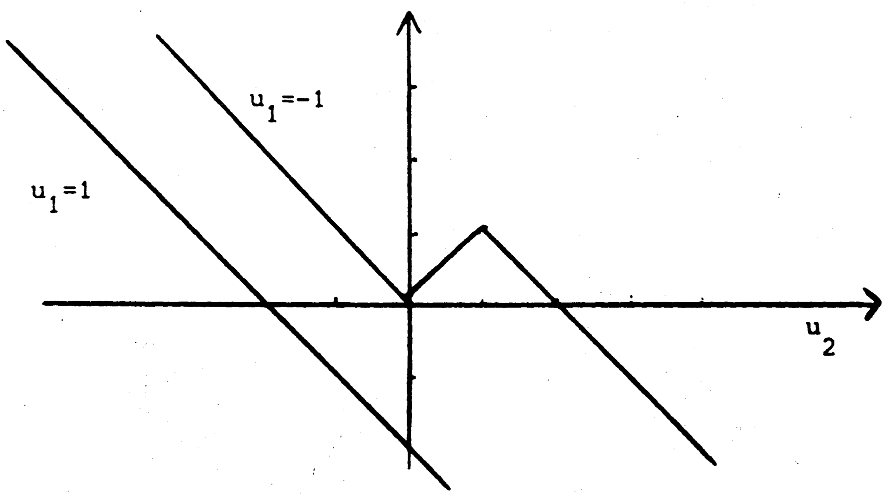

Figure 3.2. $f_2(u_1,u_2)$

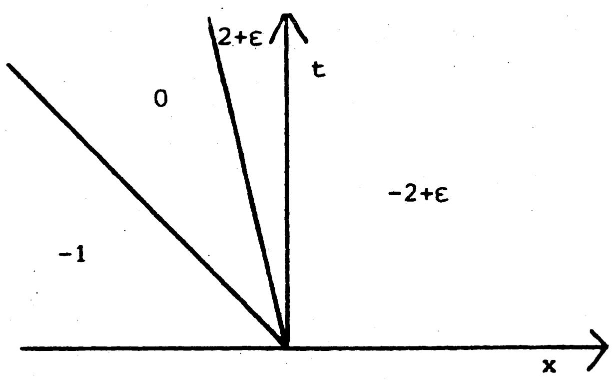

Figure 3.3. $u_2^+(x,t)$

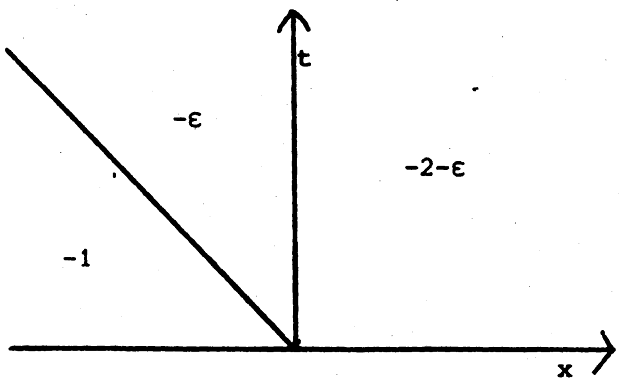

Figure 3.4. $u_2^-(x,t)$

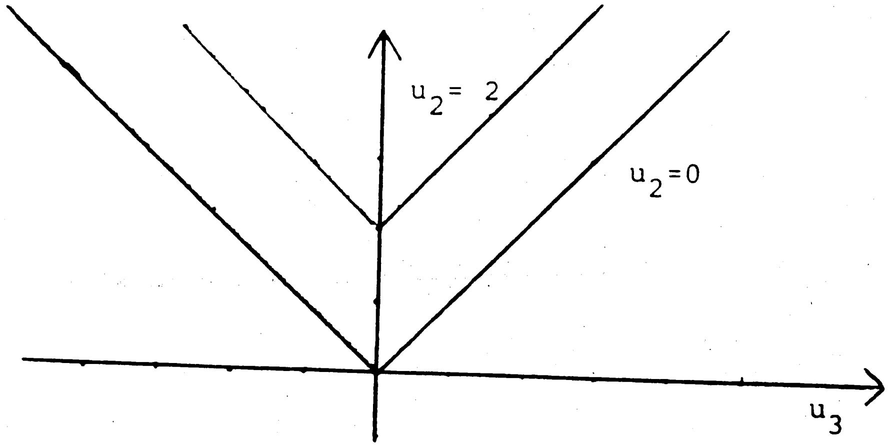

Figure 3.5. $f_3(u_2, u_3)$

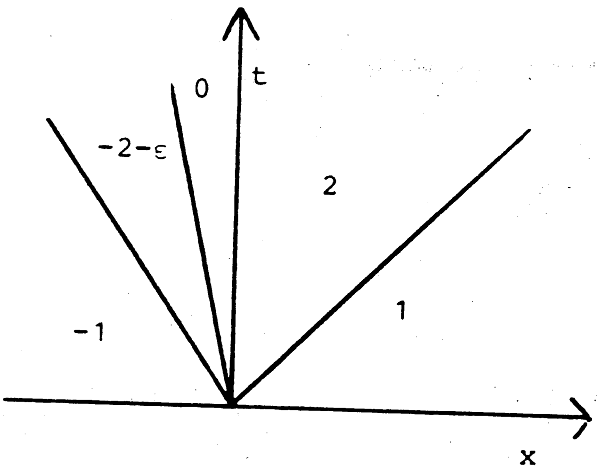

Figure 3.6. $u_3^+(x,t)$

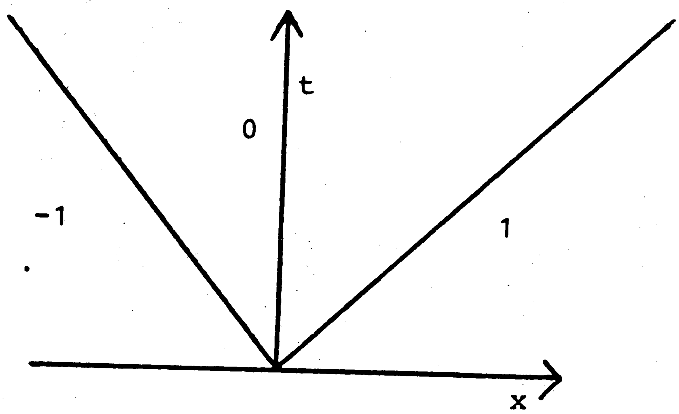

Figure 3.7. $u_3^-(x,t)$

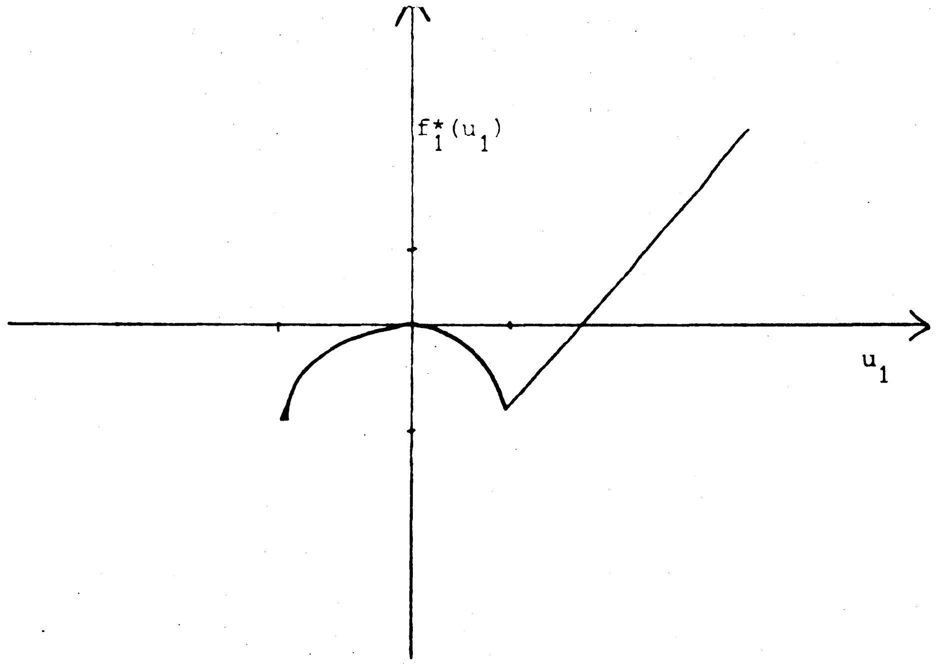

Figure 3.8

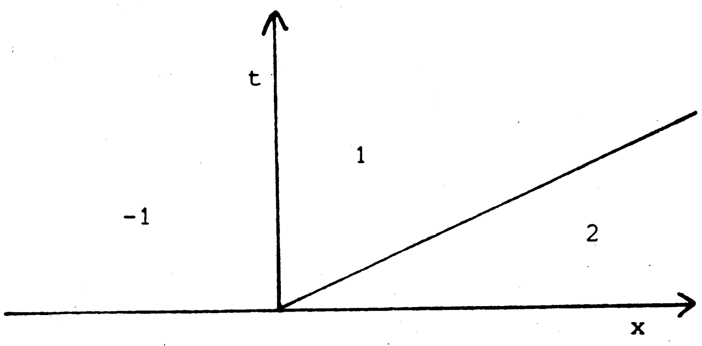

Figure 3.9 $u_1^*(x,t)$

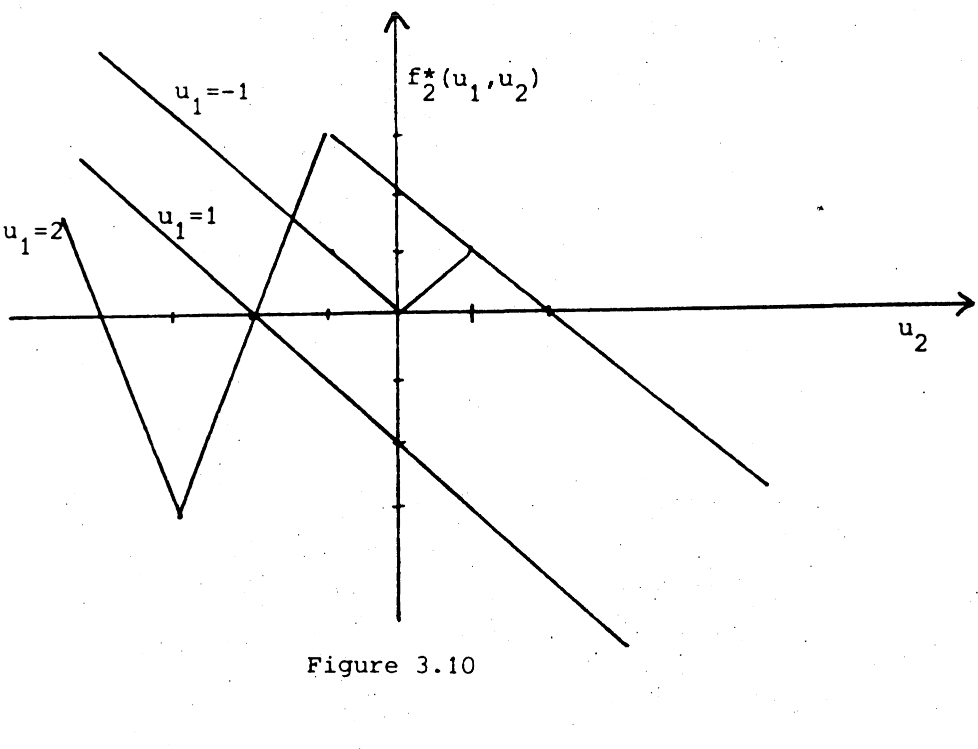

Figure 3.10